

Incidences between planes over finite fields

Nguyen Duy Phuong* Pham Van Thang[†] Le Anh Vinh[‡]

Abstract

We use methods from spectral graph theory to obtain bounds on the number of incidences between k -planes and h -planes in \mathbb{F}_q^d which generalize a recent result given by Bennett, Iosevich, and Pakianathan (2014). More precisely, we prove that the number of incidences between a set \mathcal{P} of k -planes and a set \mathcal{H} of h -planes with $h \geq 2k + 1$, which is denoted by $I(\mathcal{P}, \mathcal{H})$, satisfies

$$\left| I(\mathcal{P}, \mathcal{H}) - \frac{|\mathcal{P}||\mathcal{H}|}{q^{(d-h)(k+1)}} \right| \lesssim q^{\frac{(d-h)h+k(2h-d-k+1)}{2}} \sqrt{|\mathcal{P}||\mathcal{H}|}.$$

1 Introduction

Let \mathbb{F}_q be a finite field of q elements where q is an odd prime power. Let \mathcal{P} be a set of points, \mathcal{L} a set of lines, and $I(\mathcal{P}, \mathcal{L})$ the number of incidences between \mathcal{P} and \mathcal{L} . In [2] Bourgain, Katz, and Tao proved that the number of incidences between a point set of N points and a line set N lines is at most $O(N^{3/2-\epsilon})$. Here and throughout, $X \gtrsim Y$ means that $X \geq CY$ for some constant C and $X \gg Y$ means that $Y = o(X)$, where X, Y are viewed as functions of the parameter q .

Note that one can easily obtain the bound $N^{3/2}$ by using the Turán number and the fact that two lines intersect in at most one point. The relationship between ϵ and α in the result of Bourgain, Katz, and Tao is difficult to determine, and it is far from tight. If $N \ll q$, then Grosu [7] proved that one can embed the point set and the line set to \mathbb{C}^2 without changing the incidence structure. Thus it follows from a tight bound on the number of incidences between points and lines in \mathbb{C}^2 due to Tóth [18] that $I(\mathcal{P}, \mathcal{L}) = O(N^{4/3})$. By using methods from spectral graph theory, the third listed author [20] gave a tight bound for the case $N > q$ as follows.

*University of Science, Vietnam National University Hanoi Email: duyphuong@vnu.edu.vn

[†]EPFL, Lausanne, Switzerland. Research partially supported by Swiss National Science Foundation Grants 200020-144531 and 200021-137574. Email: thang.pham@epfl.ch

[‡]University of Education, Vietnam National University Hanoi. Research was supported by Vietnam National Foundation for Science and Technology Development grant 101.99-2013.21. Email: vinhla@vnu.edu.vn

Theorem 1.1. *Let \mathcal{P} be a set of points and \mathcal{L} be a set of lines in \mathbb{F}_q^2 . Then we have*

$$\left| I(\mathcal{P}, \mathcal{L}) - \frac{|\mathcal{P}||\mathcal{L}|}{q} \right| \leq q^{1/2} \sqrt{|\mathcal{P}||\mathcal{L}|}. \quad (1.1)$$

It follows from Theorem 1.1 that if $N \geq q^{3/2}$, then the number of incidences between points and lines is at most $(1 + o(1))N^{4/3}$, which meets the Szemerédi-Trotter bound. Theorem 1.1 has many interesting applications in several combinatorial problems, see for example [8, 9, 11, 20].

It also follows from the lower bound that if $|\mathcal{P}||\mathcal{L}| \gtrsim q^3$, then there exists at least one pair $(p, l) \in \mathcal{P} \times \mathcal{L}$ such that $p \in l$. The lower bound of Theorem 1.1 is proved to be sharp up to a constant in the sense that there exist a point set \mathcal{P} and a line set \mathcal{L} with $|\mathcal{P}| = |\mathcal{L}| = q^{3/2}$ without incidences (see [21] for more details). Furthermore, the third listed author proved that almost every point set \mathcal{P} and line set \mathcal{L} in \mathbb{F}_q^d of cardinality $|\mathcal{P}| = |\mathcal{L}| \gtrsim q$, there exists at least one incidence $(p, l) \in \mathcal{P} \times \mathcal{L}$. More precisely, the statement is as follows.

Theorem 1.2 (Vinh [21]). *For any $\alpha > 0$, there exist an integer $q_0 = q(\alpha)$ and a number $C_\alpha > 0$ satisfying the following property. When a point set \mathcal{P} and a line set \mathcal{L} with $|\mathcal{P}| = |\mathcal{L}| = s \geq C_\alpha q$, are chosen randomly in \mathbb{F}_q^2 , then the probability of $\{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\} \equiv \emptyset$ is at most α^s , provided that $q \geq q_0$.*

Using the same ideas, the third listed author [20] generalized Theorem 1.1 to the case of points and hyperplanes in \mathbb{F}_q^d as follows.

Theorem 1.3 (Vinh [20]). *Let \mathcal{P} be a set of points and \mathcal{H} be a set of hyperplanes in \mathbb{F}_q^d . Then the number of incidences between points and hyperplanes satisfies*

$$\left| I(\mathcal{P}, \mathcal{H}) - \frac{|\mathcal{P}||\mathcal{H}|}{q} \right| \leq (1 + o(1))q^{(d-1)/2} \sqrt{|\mathcal{P}||\mathcal{H}|}.$$

By using counting arguments and the upper bound on the number of incidences between points and hyperplanes, Bennett, Iosevich, and Pakianathan [1] extended Theorem 1.3 to the incidences between points and k -planes, where a k -plane is defined as follows.

Definition 1.4. *Let V be a subset in the vector space \mathbb{F}_q^d . Then V is a k -plane in \mathbb{F}_q^d , $k < d$, if there exist $k + 1$ vectors v_1, \dots, v_{k+1} in \mathbb{F}_q^d satisfying*

$$V = \text{span}\{v_1, \dots, v_k\} + v_{k+1}, \quad \text{rank}\{v_1, \dots, v_k\} = k.$$

Theorem 1.5 (Bennett et al. [1]). *Let \mathcal{P} be a set of points and \mathcal{H} be a set of k -planes in \mathbb{F}_q^d . Then there is no more than*

$$\frac{|\mathcal{P}||\mathcal{H}|}{q^{d-k}} + (1 + o(1))q^{k(d-k)/2} \sqrt{|\mathcal{P}||\mathcal{H}|}$$

incidences between the point set \mathcal{P} and the plane set \mathcal{H} .

In this paper, we will extend Theorem 1.5 to the case of k -planes and h -planes with $h \geq 2k + 1$ in the following theorem.

Theorem 1.6. *Let \mathcal{P} be a set of k -planes and \mathcal{H} be a set of h -planes in \mathbb{F}_q^d ($h \geq 2k + 1$). Then the number of incidences between \mathcal{P} and \mathcal{H} satisfies*

$$\left| I(\mathcal{P}, \mathcal{H}) - \frac{|\mathcal{P}||\mathcal{H}|}{q^{(d-h)(k+1)}} \right| \leq \sqrt{2k+1} q^{\frac{(d-h)h+k(2h-d-k+1)}{2}} \sqrt{|\mathcal{P}||\mathcal{H}|}.$$

It follows from Theorem 1.6 that if $|\mathcal{P}||\mathcal{H}| \gtrsim q^{d(k+h)+2d+k}/q^{k^2+h^2+2h}$ then the set of incidences between \mathcal{P} and \mathcal{H} is nonempty, and if $|\mathcal{P}||\mathcal{H}| \gg q^{d(k+h)+2d+k}/q^{k^2+h^2+2h}$ then $I(\mathcal{P}, \mathcal{H})$ is close to the expected number $|\mathcal{P}||\mathcal{H}|/q^{(d-h)(k+1)}$.

The study of incidence problems over finite fields received a considerable amount of attention in recent years, see for example [3, 5, 7, 10, 13, 14, 16, 17, 15, 19].

A related question that has recently received attention is the following: Given a point set \mathcal{P} in \mathbb{F}_q^2 , what is the cardinality of the set of k -rich lines, i.e. lines contain at least k points from \mathcal{P} ? Note that this question is quite different from the real case. In the real case, it follows from the Szemerédi-Trotter theorem that the number of k -rich lines determined by a set of n points is $O(n^2/k^3)$ for any $k \geq 2$, but in the finite fields case, it follows from Theorem 1.1 that the number of k -rich lines determined by a point set \mathcal{P} is at most $q|\mathcal{P}|/(k - q^{-1}|\mathcal{P}|)^2$ with $k > |\mathcal{P}|/q$. In [15], Lund and Saraf introduced an approach to deal with this problem. More precisely, they proved that, for any $k \geq 2$, there are at least cq^2 k -rich lines determined by a point set of cardinality $2(k-1)q$ for some constant $0 < c < 1$. This implies that there are at least cq^2 distinct lines determined by a set of $2q$ points. They also proved that

Theorem 1.7 (Lund-Saraf [15]). *For any integer $t \geq 2$, let \mathcal{H} be a set of the h -planes in \mathbb{F}_q^d of the cardinality*

$$|\mathcal{H}| \geq 2(t-1)q^{d-h}.$$

Then the number of points contained in at least t h -planes from \mathcal{H} is at least cq^d , where $c = (t-1)/(t-1 + 2q^{h(d-h-1)})$.

We note that in the case $h < d-1$ and $t < q^{h(d-h-1)}$, the constant c depends on q . This condition is necessary since, for instance, one can take a set of $2q^2$ lines in the union of two planes in \mathbb{F}_q^3 , then the number of 2-rich points is at most $O(q^2)$. On the other hand, it follows from Theorem 1.7 for the case $d = 3$ and $h = 1$ that the number of 2-rich points is at least $\Omega(q^2)$. This implies that the theorem is tight in this case.

Using Lund and Saraf's approach and the properties of plane-incidence graphs in Section 3, we obtain generalizations of their results as follows.

Theorem 1.8. *For any $t \geq 2$, let \mathcal{H} be a set of h -planes in vector space over \mathbb{F}_q^d with the cardinality*

$$|\mathcal{H}| \geq 2(t-1)q^{(d-h)(k+1)}.$$

Then the number of k -planes contained in at least t h -planes in \mathcal{H} is at least $cq^{(d-k)(k+1)}$, where $c = (t-1)/((t-1) + 2q^{(d-h-1)(h-k)+k})$.

Theorem 1.9. *For any $t \geq 2$, let \mathcal{K} be a set of k -planes in vector space over \mathbb{F}_q^d with the cardinality*

$$|\mathcal{K}| \geq 2(t-1)q^{(d-h)(k+1)}.$$

Then the number of h -planes containing at least t k -planes in \mathcal{K} is at least $cq^{(d-h)(h+1)}$, where $c = (t-1)/((t-1) + 2q^{k(h-k+1)})$.

2 Expander Mixing Lemma

We say that a bipartite graph is *biregular* if in both of its two parts, all vertices have the same degree. If A is one of the two parts of a bipartite graph, we write $\deg(A)$ for the common degree of the vertices in A . Label the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. Note that in a bipartite graph, we have $\lambda_2 = -\lambda_1$. The following version of the expander mixing lemma is proved in [6]. We give a detailed proof for the sake of completeness.

Lemma 2.1. *Suppose G is a bipartite graph with parts A, B such that the vertices in A all have degree a and the vertices in B all have degree b . For any two sets $X \subset A$ and $Y \subset B$, the number of edges between X and Y , denoted by $e(X, Y)$, satisfies*

$$\left| e(X, Y) - \frac{a}{|B|} |X| |Y| \right| \leq \lambda_3 \sqrt{|X| |Y|},$$

where λ_3 is the third eigenvalue of G .

Proof. We assume that the vertices of G are labeled from 1 to $|A| + |B|$. Let M be the adjacency matrix of G having the form

$$M = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix},$$

where N is the $|A| \times |B|$ matrix, and $N_{ij} = 1$ if and only if there is an edge between i and j . Firstly, let us recall some properties of eigenvalues of the matrix M . Since all vertices in A have degree a and all vertices in B have degree b , all eigenvalues of M are bounds by \sqrt{ab} . Indeed, let us denote the L_1 vector norm by $\|\cdot\|_1$, and \mathbf{e}_v an unit vector having a 1 in the position for vertex v and zeroes elsewhere. It is easy to see that $\|M^2 \cdot \mathbf{e}_v\|_1 \leq ab$. Therefore, all eigenvalues of M are bounded by \sqrt{ab} . Let $\mathbf{1}_X$ denote the column vector having 1s in the positions corresponding to the set of vertices X and 0s elsewhere. Then we have

$$M(\sqrt{a}\mathbf{1}_A + \sqrt{b}\mathbf{1}_B) = b\sqrt{a}\mathbf{1}_B + a\sqrt{b}\mathbf{1}_A = \sqrt{ab}(\sqrt{a}\mathbf{1}_A + \sqrt{b}\mathbf{1}_B),$$

$$M(\sqrt{a}\mathbf{1}_A - \sqrt{b}\mathbf{1}_B) = b\sqrt{a}\mathbf{1}_B - a\sqrt{b}\mathbf{1}_A = -\sqrt{ab}(\sqrt{a}\mathbf{1}_A - \sqrt{b}\mathbf{1}_B),$$

which implies that $\lambda_1 = \sqrt{ab}$ and $\lambda_2 = -\sqrt{ab}$ are the first and the second eigenvalues corresponding to eigenvectors $(\sqrt{a}\mathbf{1}_A + \sqrt{b}\mathbf{1}_B)$ and $(\sqrt{a}\mathbf{1}_A - \sqrt{b}\mathbf{1}_B)$, respectively.

Let W^\perp be a subspace spanned by two vectors $\mathbf{1}_A$ and $\mathbf{1}_B$. Since M is a symmetric matrix, the eigenvectors of M except $\sqrt{a}\mathbf{1}_A + \sqrt{b}\mathbf{1}_B$ and $\sqrt{a}\mathbf{1}_A - \sqrt{b}\mathbf{1}_B$ span W . Therefore, for any $u \in W$, we have $Mu \in W$, and $\|Mu\| \leq \lambda_3 \|u\|$. We have the following observations.

1. Let K be a matrix of the form $\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$, where J is the $|A| \times |B|$ all-ones matrix.

If $u \in W$, then $Ku = 0$ since every row of K is either $\mathbf{1}_A^T$ or $\mathbf{1}_B^T$.

2. If $w \in W^\perp$, then $(M - (a/|B|)K)w = 0$. Indeed, it follows from the facts that $a|A| = b|B|$, and $M\mathbf{1}_A = b\mathbf{1}_B = (a/|B|)K\mathbf{1}_A$, $M\mathbf{1}_B = a\mathbf{1}_A = (a/|B|)K\mathbf{1}_B$.

Since $e(X, Y) = \mathbf{1}_Y^T M \mathbf{1}_X$ and $|X||Y| = \mathbf{1}_Y^T K \mathbf{1}_X$,

$$\left| e(X, Y) - \frac{a}{|B|} |X||Y| \right| = \left| \mathbf{1}_Y^T (M - \frac{a}{|B|} K) \mathbf{1}_X \right|.$$

For any vector v , let \bar{v} denote the orthogonal projection onto W , so that $\bar{v} \in W$, and $v - \bar{v} \in W^\perp$. Thus

$$\mathbf{1}_Y^T (M - \frac{a}{|B|} K) \mathbf{1}_X = \mathbf{1}_Y^T (M - \frac{a}{|B|} K) \bar{\mathbf{1}}_X = \mathbf{1}_Y^T M \bar{\mathbf{1}}_X = \bar{\mathbf{1}}_Y^T M \bar{\mathbf{1}}_X.$$

Therefore,

$$\left| e(X, Y) - \frac{a}{|B|} |X||Y| \right| \leq \lambda_3 \|\bar{\mathbf{1}}_X\| \|\bar{\mathbf{1}}_Y\|.$$

Since

$$\bar{\mathbf{1}}_X = \mathbf{1}_X - ((\mathbf{1}_X \cdot \mathbf{1}_A) / (\mathbf{1}_A \cdot \mathbf{1}_A)) \mathbf{1}_A = \mathbf{1}_X - (|X|/|A|) \mathbf{1}_A,$$

we have $\|\bar{\mathbf{1}}_X\| = \sqrt{|X|(1 - |X|/|A|)}$. Similarly, $\|\bar{\mathbf{1}}_Y\| = \sqrt{|Y|(1 - |Y|/|B|)}$.

In other words,

$$\left| e(X, Y) - \frac{a}{|B|} |X||Y| \right| \leq \lambda_3 \sqrt{|X||Y|(1 - |X|/|A|)(1 - |Y|/|B|)},$$

and the lemma follows. \square

Lemma 2.2. *Let G be a biregular graph with parts A, B and $|A| = m$, $|B| = n$. We label vertices of G from 1 to $|A| + |B|$. Let M be the adjacency matrix of G having the form*

$$M = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix},$$

where N is the $|A| \times |B|$ matrix, and $N_{ij} = 1$ if and only if there is an edge between i and j . Let $v_3 = (v_1, \dots, v_m, u_1, \dots, u_n)$ be an eigenvector of M corresponding to the eigenvalue λ_3 . Then we have (v_1, \dots, v_m) is an eigenvector of NN^T , and $J(v_1, \dots, v_m) = 0$, where J is the $m \times m$ all-ones matrix.

Proof. We have

$$M^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^T N \end{pmatrix}.$$

Since v_3 is an eigenvector of M with eigenvalue λ_3 , v_3 is also an eigenvector of M^2 with the eigenvalue λ_3^2 . On the other hand,

$$M^2 v_3 = \begin{pmatrix} NN^T \cdot (v_1, \dots, v_m)^T \\ N^T N \cdot (u_1, \dots, u_n)^T \end{pmatrix} = \lambda_3^2 (v_1, \dots, v_m, u_1, \dots, u_n)^T.$$

This implies that (v_1, \dots, v_m) is an eigenvector of NN^T corresponding to the eigenvalue λ_3^2 . We also note that it follows from proof of Lemma 2.1 that if v_3 is an eigenvector corresponding the third eigenvalue of M , then $Kv_3 = 0$, which implies that $J(v_1, \dots, v_m) = 0$. We also note that λ_3^2 is the second eigenvalue of NN^T . \square

It follows from Lemma 2.2 that in order to bound the third eigenvalue of M^2 , it suffices to bound the second eigenvalue of the matrix NN^T .

Suppose that $G = (A, B, E)$ is a bipartite graph as in Lemma 2.1. For any set S of vertices in A , we denote the set of vertices in B that have at least t neighbors in S by $R_t(S)$. Similarly, we have the definition of $R_t(S)$ with $S \subset B$. In [15], Lund and Saraf proved the following theorem.

Theorem 2.3. *If a set S of vertices in B such that $|S| \geq 2(t-1)|B|/\deg(A)$, then $|R_t(S)| \geq c|A|$, where $c = (t-1)/(t-1+2\deg(A)\mu^2)$, $\mu = \lambda_3/\lambda_1$.*

3 Plane-incidence graphs

We now construct the plane-incidence graph $G_P = (A, B, E)$ as follows. The first vertex part is the set of all k -planes, and the second vertex part is the set of all h -planes. There is an edge between a k -plane v and a h -plane p if v lies on p . It is easy to check that

$$|A| = \frac{q^d(q^d-1)\cdots(q^d-q^{k-1})}{q^k(q^k-1)\cdots(q^k-q^{k-1})}, \quad |B| = \frac{q^d(q^d-1)\cdots(q^d-q^{h-1})}{q^h(q^h-1)\cdots(q^h-q^{h-1})}.$$

Now, we will count the degree of each vertex of the graph G_P . We first need the following lemmas.

Lemma 3.1. *Let $U = \text{span}\{u_1, \dots, u_k\} + u_{k+1}$ and $V = \text{span}\{v_1, \dots, v_k\} + v_{k+1}$ be two k -planes in \mathbb{F}_q^d . Then $U \equiv V$ if and only if $\text{span}\{u_1, \dots, u_k\} \equiv \text{span}\{v_1, \dots, v_k\}$ and $u_{k+1} \in V, v_{k+1} \in U$.*

Proof. If $\text{span}\{u_1, \dots, u_k\} \equiv \text{span}\{v_1, \dots, v_k\}$ and $u_{k+1} \in V, v_{k+1} \in U$, then it is easy to check that $U \equiv V$. For the inverse case, if $U \equiv V$, then $u_{k+1} \in V$ and $v_{k+1} \in U$. We need to prove that $\text{span}\{u_1, \dots, u_k\} \equiv \text{span}\{v_1, \dots, v_k\}$. Indeed, without loss of generality, we assume that there exists an element u_i for some $1 \leq i \leq k$ such that $u_i \notin \text{span}\{v_1, \dots, v_k\}$, then we will prove that this leads to a contradiction. Since $U \equiv V$, $u_i + u_{k+1} \in V$. Therefore, there exist elements $a_1, \dots, a_k \in \mathbb{F}_q$ such that $u_i + u_{k+1} = \sum_{j=1}^k a_j v_j + v_{k+1}$. On the other hand, since $u_{k+1} \in V$, there exist elements $b_1, \dots, b_k \in \mathbb{F}_q$ such that $u_{k+1} = \sum_{j=1}^k b_j v_j + v_{k+1}$. This implies that $u_i = \sum_{j=1}^k (a_j - b_j) v_j$, which leads to a contradiction. \square

Lemma 3.2. *Let $U = \text{span}\{u_1, \dots, u_k\} + u_{k+1}$, $V = \text{span}\{v_1, \dots, v_k\} + v_{k+1}$ be two k -planes in \mathbb{F}_q^d . For any $h > k$, if the h -plane $H = \text{span}\{t_1, \dots, t_h\} + t_{h+1}$ contains both of them then H can be written as $H = \text{span}\{t_1, \dots, t_h\} + u_{k+1}$ and $u_1, \dots, u_k, v_1, \dots, v_k, u_{k+1} - v_{k+1} \in \text{span}\{t_1, \dots, t_h\}$.*

Proof. First we need to prove that for any vector $x \in H$, H can be written as $H = \text{span}\{t_1, \dots, t_h\} + x$. Indeed, since $x \in H$, x can be presented as $x = \sum_{i=1}^h a_i t_i + t_{h+1}$ with $a_i \in \mathbb{F}_q$. Let $y = \sum_{i=1}^h b_i t_i + t_{h+1}$ be a vector in H , then y can also be written as $y = \sum_{i=1}^h (b_i - a_i) t_i + x$. This implies that $y \in \text{span}\{t_1, \dots, t_h\} + x$. The inverse case $\text{span}\{t_1, \dots, t_h\} + x \subset H$ is trivial.

If H contains both U and V , then H can be presented as $H = \text{span}\{t_1, \dots, t_h\} + u_{k+1}$ since $u_{k+1} \in H$. It is easy to see that $u_i \in \text{span}\{t_1, \dots, t_h\}$ for all $1 \leq i \leq k$. Since V is contained in H , $v_{k+1} \in H$, which implies that $v_{k+1} - u_{k+1} \in \text{span}\{t_1, \dots, t_h\}$, and $v_i \in \text{span}\{t_1, \dots, t_h\}$ for all $1 \leq i \leq k$. \square

Using Lemma 3.1 and Lemma 3.2, we obtain that the degree of each h -plane is

$$\frac{q^h}{q^k} \prod_{i=0}^{k-1} \frac{q^h - q^i}{q^k - q^i} = (1 + o(1)) q^{(h-k)(k+1)}.$$

In order to count the degree of each k -plane, we will use similar arguments as in the proof of [1, Theorem 2.3]. Let $x(h, k)$ be the number of distinct k -planes in a h -plane. Let $y(h, k)$ be the number of distinct h -planes in \mathbb{F}_q^d containing a fixed k -plane. Then we have

$$y(h, k) = \frac{x(h, k)x(d, h)}{x(d, k)}.$$

On the other hand, we just proved that

$$x(h, k) = \frac{q^h}{q^k} \prod_{i=0}^{k-1} \frac{q^h - q^i}{q^k - q^i},$$

which implies that

$$y(h, k) = \prod_{i=k}^{h-1} \frac{q^{d-i} - 1}{q^{h-i} - 1} = (1 + o(1)) q^{(d-h)(h-k)}.$$

In short, the degree of each k -plane is $(1 + o(1)) q^{(d-h)(h-k)}$. We are now ready to bound the third eigenvalue of M in the following lemma.

Lemma 3.3. *The third eigenvalue of M is bounded by $q^{((d-h)h+k(2h-d-k+1))/2}$.*

Proof. Let M be the adjacency matrix of G_P , which has the form

$$M = \begin{bmatrix} 0 & N \\ N^T & 0 \end{bmatrix},$$

where N is a $q^{(d-k)(k+1)} \times q^{(d-h)(h+1)}$ matrix, and $N_{vp} = 1$ if $v \in p$, and zero otherwise. Therefore,

$$M^2 = \begin{bmatrix} NN^T & 0 \\ 0 & N^T N \end{bmatrix}.$$

It follows from Lemma 2.2 that it suffices to bound the second eigenvalue of NN^T . Given any two k -planes $V_1 = \text{span}\{u_1, \dots, u_k\} + u_{k+1}$ and $V_2 = \text{span}\{v_1, \dots, v_k\} + v_{k+1}$, we now count the number of their common neighbors, i.e. the number of h -planes containing both of them. We assume that $H = \text{span}\{t_1, \dots, t_h\} + t_{h+1}$ is a h -plane supporting V_1 and V_2 . Then it follows from Lemma 3.1 and Lemma 3.2 that H can be written as $H = \text{span}\{t_1, \dots, t_h\} + u_{k+1}$ and $u_1, \dots, u_k, v_1, \dots, v_k, u_{k+1} - v_{k+1} \in \text{span}\{t_1, \dots, t_h\}$. Thus the number of h -planes supporting V_1 and V_2 depends on the rank of the following system of vectors $\text{rank}(V_1, V_2) := \{u_1, \dots, u_k, v_1, \dots, v_k, v_{k+1} - u_{k+1}\}$. We also note that $k+1 \leq \text{rank}(V_1, V_2) \leq 2k+1$ since V_1 and V_2 are distinct. We assume that $\text{rank}(V_1, V_2) = t$ and $\text{span}\{u_1, \dots, u_k, v_1, \dots, v_k, v_{k+1} - u_{k+1}\} \equiv \text{span}\{w_1, \dots, w_t\}$ with $w_i \in \mathbb{F}_q^d$ for $1 \leq i \leq t$, then the number of h -planes containing both V_1 and V_2 equals the number of $(h-t)$ -tuples of vectors $\{x_1, \dots, x_{h-t}\}$ in \mathbb{F}_q^d such that $\text{rank}\{w_1, \dots, w_t, x_1, \dots, x_{h-t}\} = h$. Thus the number of common neighbors of V_1 and V_2 is

$$\frac{(q^d - q^t)(q^d - q^{t+1}) \dots (q^d - q^{h-1})}{(q^h - q^t) \dots (q^h - q^{h-1})} = (1 + o(1))q^{(d-h)(h-t)}.$$

Therefore, NN^T can be presented as

$$\begin{aligned} NN^T &= q^{(d-h)(h-2k-1)} J + (q^{(d-h)(h-k)} - q^{(d-h)(h-2k-1)}) I \\ &+ \sum_{k+1 \leq t \leq 2k} (q^{(d-h)(h-t)} - q^{(d-h)(h-2k-1)}) E_t \end{aligned} \quad (3.1)$$

where I is the identity matrix and J is the all-one matrix, and for each t , E_t are the adjacency matrix of the graphs $G(E_t)$: $V(G(E_t))$ is the set of all k -planes, and there is an edge between two k -planes $V_1 = \text{span}\{u_1, \dots, u_k\} + u_{k+1}$ and $V_2 = \text{span}\{v_1, \dots, v_k\} + v_{k+1}$ if and only if $\text{rank}(V_1, V_2) = t$. We note that these graphs are regular, and we count their degree as follows. For each $k+1 \leq t \leq 2k$, and each vertex V_1 , we now count the number of k -planes V_2 such that $\text{rank}(V_1, V_2) = t$. In order to that, we consider two following cases

1. If $u_{k+1} - v_{k+1} \in \text{span}\{u_1, \dots, u_k, v_1, \dots, v_k\}$, then the number of V_2 is

$$\frac{k!q^t(q^d - q^k) \dots (q^d - q^{t-1})(q^t - q^{t-k}) \dots (q^t - q^{k-1})}{(t-k)!(2k-t)!q^k(q^k - 1) \dots (q^k - q^{k-1})} = (1 + o(1))q^{(t-k)(d-t+k+1)},$$

where $(q^d - q^k) \dots (q^d - q^{t-1})/(t-k)!$ is the number of $(t-k)$ -tuples $\{v_1, \dots, v_{t-k}\}$ such that $\text{rank}\{u_1, \dots, u_k, v_1, \dots, v_{t-k}\} = t$, and $(q^t - q^{t-k}) \dots (q^t - q^{k-1})/(2k-t)!$ is the number of $(2k-t)$ -tuples of vectors $\{v_{t-k+1}, \dots, v_k\}$ in $\text{span}\{u_1, \dots, u_k, v_1, \dots, v_{t-k}\}$ such that $\text{rank}\{v_1, \dots, v_k\} = k$, and q^t is the number of choices of $u_{k+1} - v_{k+1}$ in

$\text{span}\{u_1, \dots, u_k, v_1, \dots, v_k\}$, the term $q^k(q^k - 1) \dots (q^k - q^{k-1})/k!$ is the number of different ways to present a k -plane. We note that for each choice of $u_{k-1} - v_{k-1}$, then v_{k+1} is determined uniquely.

2. If $u_{k+1} - v_{k+1} \notin \text{span}\{u_1, \dots, u_k, v_1, \dots, v_k\}$, then the number of V_2 is

$$\frac{k!(q^d - q^k) \dots (q^d - q^{t-1})(q^{t-1} - q^{t-k-1}) \dots (q^{t-1} - q^{k-1})}{(t-k)!(2k-t+1)!q^k(q^k - 1) \dots (q^k - q^{k-1})} = (1+o(1))q^{(t-k)(d-t+k+2)-1-k},$$

where $(q^d - q^k) \dots (q^d - q^{t-1})/(t-k)!$ is the number of $(t-k)$ -tuples $\{v_1, \dots, v_{t-k-1}, u_{k+1} - v_{k+1}\}$ such that $\text{rank}\{u_1, \dots, u_k, v_1, \dots, v_{t-k-1}, u_{k+1} - v_{k+1}\} = t$, and the second term $(q^{t-1} - q^{t-k-1}) \dots (q^{t-1} - q^{k-1})/(2k-t+1)!$ is the number of $(2k-t+1)$ -tuples $\{v_{t-k}, \dots, v_k\}$ in $\text{span}\{u_1, \dots, u_k, v_1, \dots, v_{t-k-1}\}$ such that $\text{rank}\{v_1, \dots, v_k\} = k$.

Therefore, for each t , the degree of any vertex in $V(G(E_t))$ is

$$(1 + o(1)) (q^{(t-k)(d-t+k+1)} + q^{(t-k)(d-t+k+2)-1-k}) = (1 + o(1))q^{(t-k)(d-t+k+1)}.$$

Let $v_3 = (v_1, \dots, v_{|A|}, u_1, \dots, u_{|B|})$ be the third eigenvector of M^2 . Lemma 2.2 implies that $(v_1, \dots, v_{|A|})$ is an eigenvector of NN^T corresponding to the eigenvalue λ_3^2 . It follows from the equation (3.1) that

$$(\lambda_3^2 - (q^{(d-h)(h-k)} - q^{(d-h)(h-2k-1)})) v_3 = \left(\sum_{k+1 \leq t \leq 2k} (q^{(d-h)(h-t)} - q^{(d-h)(h-2k-1)}) E_t \right) v_3.$$

Hence, v_3 is an eigenvector of

$$\sum_{k+1 \leq t \leq 2k} (q^{(d-h)(h-t)} - q^{(d-h)(h-2k-1)}) E_t.$$

Since eigenvalues of sum of matrices are bounded by sum of largest eigenvalues of the summands, we have

$$\lambda_3^2 \leq q^{(d-h)(h-k)} + q^{(d-h)(h-2k-1)} + kq^{(d-h)h+k(2h-d-k+1)} \leq (2k+1)q^{(d-h)h+k(2h-d-k+1)},$$

which completes the proof of lemma. \square

4 Proofs of Theorems 1.6, 1.8, and 1.9

Proof of Theorem 1.6. Since the degree of each k -plane is $(1 + o(1))q^{(d-h)(h-k)}$, and the number of h -planes is $(1 + o(1))q^{(d-h)(h+1)}$, we have

$$\frac{\deg(A)}{|B|} = (1 + o(1))q^{-(d-h)(k+1)}.$$

Thus, Theorem 1.6 follows by combining Lemma 2.1 and Lemma 3.3. \square

Proof of Theorems 1.8 and 1.9. Combining Theorem 2.3 and Lemma 3.3, Theorem 1.8 and Theorem 1.9 follow. \square

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